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ARTICLE

Noûs

Arithmetical pluralism and the objectivity of syntax

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Abstract
Arithmetical pluralism is the view that there is not one true arithmetic but rather many apparently conflicting arithmetical theories, each true in its own language. While pluralism has recently attracted considerable interest, it has also faced significant criticism. One powerful objection, which can be extracted from Parsons (2008), appeals to a categoricity result to argue against the possibility of seemingly conflicting true arithmetics. Another salient objection raised by Putnam (1994) and Koellner (2009) draws upon the arithmetization of syntax to argue that arithmetical pluralism is inconsistent with the objectivity of syntax. First, we review these arguments and explain why they ultimately fail. We then offer a novel, more sophisticated argument that avoids the pitfalls of both. Our argument combines strategies from both objections to show that pluralism about arithmetic entails pluralism about syntax. Finally, we explore the viability of pluralism in light of our argument and conclude that a stable pluralist position is coherent. This position allows for the possibility of rival packages of arithmetic and syntax theories, provided that they systematically co-vary with one another.

1 INTRODUCTION

Pluralism about some area of mathematics is, roughly, the view that there is no one true theory of that domain, but many apparently conflicting theories, each of which is true in its own language.[1] Pluralism about geometry has been popular, even mainstream, ever since the discovery and systematic investigation of non-Euclidean geometry. Despite 'disagreeing' over Euclid's fifth postulate, both Euclidean and non-Euclidean geometries can be seen to be true in their own languages, relative to distinctive notions of *point* and *line*. This paper is about the prospect of pluralism about *arithmetic*, the theory of the natural numbers.

At first, this seems barely coherent. We ordinarily think we have a very clear conception of the natural numbers – so clear that any well-posed question about them has a definite answer. Are there infinitely many prime numbers that differ by 2, as the twin prime conjecture claims? Intuitively, there must be an answer – one that we would discover if we simply ran through the numbers, checking one by one. Of course, there are infinitely many numbers, so we cannot actually run through them; but this looks like a medical impossibility, not a mathematical one. And while we can make sense of someone who speaks a non-Euclidean language and sincerely and literally says things like "parallel lines sometimes intersect", it's pre-theoretically hard to see what a 'non-Euclidean' arithmetic could look like; anyone who goes around uttering counter-arithmetical sentences like "2+2 = 5" seems simply to be using the ingredient symbols in a radically different way.

Nevertheless, arithmetical pluralism has more going for it than the remarks above might suggest. Suppose some theory, A, faithfully codifies our arithmetical practice. Considerations concerning the computational and cognitive limitations of our physically-implemented minds strongly suggest that A must be recursively axiomatizable. Assuming additionally that A is consistent, it follows by Gödel's incompleteness theorems that A is also incomplete – there are statements left undecided (neither provable nor refutable) by the theory. Anti-pluralist views will insist that this is a mere epistemological problem: such statements have determinate truth-values, whether or not we can know what they are. But this leads to a serious challenge: in explaining the content of mathematical language, we are plausibly constrained to appeal to facts about its use, since we cannot cite any sort of physical interaction between us and the subject-matter that might help to fix reference or determine content.[2] So, on this view, all we can appeal to in explaining the content of mathematical language are the mathematical *theories* – broadly understood – that we employ. If so, there is pressure to think that, for claims undecided by our best theory, we are dealing with a species of *semantic* indeterminacy: there is no fact of the matter which of them or their negations are true in our language.[3] And so we are arguably led to a species of pluralism: if A doesn't decide φ, it can consistently be extended with either φ or with $\neg\varphi$, so it is plausible to

[1] For some recent discussions of pluralism and related issues, see Balaguer (1998), Hamkins (2012), Warren (2015), Warren (2020), Clarke-Doane (2020b), Clarke-Doane (2022a), Zalta (2023). Some characterizations of pluralism include a metaphysical claim that the true but apparently incompatible theories are "equally good from any objective point of view" or of equal "metaphysical merit" (cf. Warren (2015), Hirsch and Warren (2019)), but nothing in the arguments below turns on this further claim. Thanks to a referee for prompting clarification here.

[2] See Putnam (1980), Field (2001, Ch 12), Button and Walsh (2018), Warren and Waxman (2020); though see Clarke-Doane (2022a, §2.2) for a contrary suggestion that mathematical explanations are sometimes in fact causal.

[3] This sort of semantic indeterminacy should be distinguished from both 'metaphysical' indeterminacy, according to which there is indeterminacy in mathematical reality itself (i.e. the structure of the natural numbers is itself indeterminate), and from Benacerraf-style semantic indeterminacy arising from the apparent fact that mathematical practice is silent on 'extra-structural' questions like whether numbers are identical to certain particular sets (cf. Benacerraf (1965)).

regard both extensions as acceptable 'precisifications' of our, *ex hypothesi*, indeterminate practice – each true relative to its own precisified notions.[4]

Arithmetical pluralism also arises on certain 'big picture' views in the philosophy of mathematics. Plenitudinous platonists think that mathematical ontology is so plentiful – and the metasemantics of mathematical language so cooperative – that "whenever you have a consistent theory of pure mathematics [...] there are mathematical objects that satisfy that theory *under a perfectly standard satisfaction relation*".[5] And for conventionalists the truth of mathematical claims is entirely explained by linguistic conventions.[6] This leads quite naturally to pluralism: there are many possible linguistic conventions that might have been adopted, each of whose adherents would be speaking truly in their own language, governed by these alternative conventions.

As stated, plenitudinous platonism and conventionalism support a general *mathematical* pluralism; pluralism about *arithmetic* in particular doesn't immediately follow. In fact, there are ways of resisting the transition. On the plenitudinous view, for instance, even if each consistent theory correctly describes its own part of the mathematical universe, it might be argued that only one enjoys the honorific "arithmetic"; so there is space to be a monist about arithmetic while being a pluralist about mathematics. Indeed, this is Balaguer's own position: he thinks that arithmetical questions have determinate answers, because "our conception of the natural numbers is categorical [...] the totality of our natural-number thoughts picks out a unique model, or at worst, a unique class of models that are isomorphic to one another" (Balaguer, 1998, p. 64). A similar move might be made by conventionalists: perhaps there is enough determinacy built into our concept of the natural numbers that no alternative practices or conventions would really count as arithmetical; even if adherents speak truly in their own language, they would be changing the subject.

So we must address the question of what it is for a theory to be *arithmetical*. Some apparently straightforward characterizations that might seem open to anti-pluralists are problematic in a pluralist context. For instance, it's awkward to try to characterize an arithmetical theory as one whose subject matter is *the* natural numbers; for if pluralism is correct, then each of the true, apparently conflicting arithmetical theories has an equally good claim to describe 'the' natural numbers. Nor can arithmetical theories be characterized as those stated in one of the usual (natural or formal) arithmetical languages. This is neither necessary nor sufficient: not necessary, because arithmetic can at least in principle be carried out in a wide variety of languages other than those actually used; and not sufficient, because of the conventionality of signs – a community of speakers might use 'arithmetical' language, syntactically individuated, to express something entirely different.

[4] We say "arguably" because the recent literature on determinacy is rich and no clear consensus has yet formed whether our arithmetical practice is in fact determinate. Kreisel (1969) argues that it is, appealing to the philosophical significance of second-order categoricity theorems in this connection; Weston (1976), Putnam (1980), and Field (2001, postscript to Ch. 12) argue (in our view convincingly) to the contrary. More recently Parsons (2008) and Button and Walsh (2018) have argued that various kinds of *internal* categoricity results can play a similar role; see Field (2001, postscript to Ch. 12), Warren (2020), and Picollo and Waxman (2023) for arguments to the contrary (and Fischer and Zicchetti (2023) for a discussion of the role of truth in internal categoricity arguments). Warren (2021) argues for the determinacy of arithmetic on different grounds, involving our supposedly actual ability to follow infinitary rules.

[5] The canonical statement of plenitudinous platonism is Balaguer (1998). The quote is from Field (2001, p. 333), though Field does not endorse the view.

[6] Cf. Carnap (1934), Warren (2020).

A better approach is to appeal to similarity of *conceptual* or *inferential* role. Arithmetical concepts and expressions, as we use them, are governed by certain rules; we can think of their conceptual role as the uses "that are sanctioned by the complete package of rules for the language" (Warren, 2015, p. 1366). An *arithmetical* theory, even perhaps expressed in an alien language, is then characterized as such by having a conceptual role that is sufficiently similar to the conceptual role that arithmetic plays for us. This resolves the issues previously raised: it counts theories expressed in syntactically diverse languages as arithmetical; and it rules out theories which, despite being formulated in the 'right' language, play a disparate conceptual role. It also nicely illuminates the case of geometric pluralism. Why are both Euclidean and hyperbolic geometries *geometries*? Because the conceptual roles of their respective notions of *points* and *lines* are sufficiently similar, in virtue of being governed by the first four of Euclid's axioms, despite their evident differences.

What exactly is the conceptual role of arithmetic? For now, we will make minimal assumptions. It contains a pure component, which (for convenience) we will assume throughout is Peano Arithmetic (PA) or a notational variant thereof, although everything we say applies equally to other recursively enumerable theories extending PA.[7] It also includes an applied component, governing the use of arithmetical language outside of purely arithmetical contexts, e.g. in counting concrete things. This applied component will play a crucial role in our argument, but we will defer detailed discussion until when it's needed.

There are two powerful objections against arithmetical pluralism in the literature: one, which can be extracted from Parsons (2008), uses a categoricity result to argue that no two theories that satisfy the conceptual role of arithmetic can conflict; the other, by Putnam (1994) and Koellner (2009), uses the arithmetization of syntax to argue that arithmetical pluralism is incompatible with the objectivity of syntax (which they endorse). In Sections 2 and 3 we review these arguments and explain why they fail. Our main aim, however, is to offer a novel, more sophisticated argument, combining both strategies – one which is not susceptible to any of the previous objections. This new argument, which we develop in Section 4, establishes that pluralism about arithmetic entails pluralism about syntax, just like Putnam and Koellner's. But, unlike them, we do not take this to be a conclusive refutation of the view. Instead, in the final Section 5 we explore the viability of pluralism in light of our argument. We end by suggesting that there is a stable pluralist position that allows the possibility of rival *packages* of arithmetic and syntax theories, provided that they systematically co-vary with one another (in a sense to be explained).

[7] As a referee rightly points out, our claim that arithmetical theories must extend PA excludes various strict finitist views about the natural numbers (involving claims that are inconsistent with the Peano axioms; for instance, that the numbers aren't closed under multiplication, or that some number lacks a successor, or that exponentiation isn't a total function; cf. Yessenin-Volpin (1970), Nelson (1986)). Should arithmetical pluralism extend to such strict finitist views? We're sympathetic to the 'hard-line' response that strict finitist theories shouldn't be viewed as fully and genuinely arithmetical: in particular, there's a strong case that (open-ended) induction is essential for a theory to play the conceptual role of arithmetic (cf. our discussion of applications in Section 4), whereas strict finitists typically reject instances of induction even for sentences in the language of arithmetic. (For views according to which the axioms of PA are, roughly, analytic of the notion of natural number, see Dummett (1978, p. 186), Parsons (1992, p. 155), Hale and Wright (2001), Warren (2020, §8.2); though see Clarke-Doane (2022a, §1.3) for scepticism about the analyticity of mathematical axioms.) However, even if arithmetical pluralism extends to strict finitism, our discussion below can be read as addressing the interesting and controversial issue of whether arithmetical pluralism is tenable *for theories extending* PA, which we take it is precisely the claim motivated by the considerations in favour of pluralism mentioned above. The arguments of Parsons and Putnam and Koellner that we discuss in Sections 2 and 3 can be reconceived as establishing that PA is an upper bound on the scope of arithmetical pluralism, i.e. that there cannot be true but apparently incompatible arithmetical theories extending PA.

2 | PARSONS' CATEGORICITY ARGUMENT

In *Mathematical Thought and Its Objects*, Parsons gives an intricate and influential argument for the claim that "the natural numbers are at least determinate up to isomorphism: If two structures answer equally well to our conception of the sequence of natural numbers, they are isomorphic" (Parsons, 2008, p. 272). His discussion contains many subtleties, but for our purposes we can view the argument as directed against the coherence of arithmetical pluralism.

The argument appeals to a version of Dedekind's categoricity result for arithmetic.[8] While Dedekind's original result implicitly assumes second-order resources, Parsons remarks that it is "essentially first-order" and extends it to first-order arithmetic in a more-or-less straightforward manner. The simplest presentation supposes that we have two purely arithmetical (first-order) theories, T_α and T_β, stated in disjoint signatures, $\langle N_\alpha, s_\alpha, 0_\alpha, +_\alpha, \times_\alpha \rangle$ and $\langle N_\beta, s_\beta, 0_\beta, +_\beta, \times_\beta \rangle$. T_α and T_β each include a copy of the axioms of PA in the relevant signature; their axioms are appropriately relativized to N_α and N_β respectively; and, in each case, their induction schemata are understood 'open-endedly' (i.e. ranging over *all* vocabulary of the language, including possible future expansions).

The proof works in an extension, T, of $T_\alpha + T_\beta$ that contains a function-symbol f satisfying the following defining equations:[9]

$$f(0_\alpha) = 0_\beta$$
$$f(s_\alpha x) = s_\beta(f(x))$$

As N_α sustains induction over the whole language of the extended theory T, f is provably well defined. What's more, we can show in T that $f : N_\alpha \to N_\beta$ is both one-one and onto. Proving the one-one step – that $\forall x \forall y\, (N_\alpha(x) \wedge N_\alpha(y) \to (f(x) = f(y) \to x = y))$ – involves an application of T_α's induction schema to a formula containing f.[10] Proving the onto step – that $\forall x\, (N_\beta(x) \to \exists y\, (N_\alpha(y) \wedge f(y) = x))$ – involves an application of T_β's induction schema to a formula containing both f *and* N_α.[11]

We can go on to show in T that f is actually an *internal isomorphism* between T_α and T_β. That is, f is not only a bijection but also preserves 'arithmetical structure', as it satisfies the recursion equations above as well as the following:

$$f(x +_\alpha y) = f(x) +_\beta f(y)$$
$$f(x \times_\alpha y) = f(x) \times_\beta f(y)$$

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[8] Cf. Dedekind (1996).

[9] For the sake of definiteness, we may add that f maps every non-α-number to 0_β. Because of the way that f 'crosses between' the two copies of arithmetic, it doesn't strictly follow from $T_\alpha + T_\beta$ that such a function exists. Technically, this can be resolved by simply adding the relevant recursion equations as axioms. Philosophically, the additional resources required are easy to justify; as Parsons (2008, p. 281) observes, introducing functions (such as f) by primitive recursion on the natural numbers is a standard part of arithmetical practice. See Maddy and Väänänen (2024, §5.2) for more on the technical details.

[10] The induction is on the open formula $\forall y\, (N_\alpha(y) \to (f(x) = f(y) \to x = y))$. The proof makes an essential appeal to facts about s_β entailed by the basic axioms of T_β.

[11] The induction is now on the open formula $\exists y\, (N_\alpha(y) \wedge f(y) = x)$.

This result is sometimes referred to as the "internal categoricity" of first-order Peano arithmetic, as the only substantial assumption we've made about T_α and T_β is that they extend PA.[12] An immediate corollary is that T_α and T_β agree on every statement. More precisely, for each sentence φ of the language of T_α in which all quantifiers are relativized to N_α, it follows that:

$$T \vdash \varphi \leftrightarrow \varphi^\beta$$

where φ^β is the result of replacing in φ all primitive symbols by their respective 'copies' in the signature of T_β. This corollary is sometimes referred to as the "internal intolerance" of first-order Peano arithmetic.

What philosophical conclusions can be drawn? As stated, the internal categoricity result takes place in a single language that combines the vocabulary of T_α and T_β. So an *intra-linguistic* application of it yields the conclusion that anyone who speaks a language with two copies of arithmetical vocabulary is bound to view them as internally isomorphic. It is therefore inconsistent to accept two seemingly incompatible arithmetical theories (e.g. $T_\alpha := (\mathsf{PA} + \varphi)^\alpha$ and $T_\beta := (\mathsf{PA} + \neg\varphi)^\beta$, for some sentence φ undecidable in PA), even if they are expressed in disjoint signatures.

Does this have any bearing on arithmetical pluralism? Not obviously. Pluralism says that there are apparently incompatible arithmetical theories *each of which is true in its own language*. But nothing whatsoever in the view commits it to saying that these theories can or should be *combined in a single language*. At best, then, this version of categoricity tells us that pluralism is a phenomenon that can only arise across languages, not within them.

Parsons, however, attempts to put the theorem to use in an *inter-linguistic* context. Imagine that the two arithmetical theories are no longer formulated in the same language, but rather there are two speakers – Alpha and Beta – each of whom accepts one of the theories *in their own language*. Can the internal categoricity result be used to argue that Alpha and Beta must regard the other's 'numbers' as (internally) isomorphic to theirs? Here things become subtle because we have to take care over how Alpha and Beta translate each other's vocabulary and how these translations behave. We will consider Alpha's position, though the situation is perfectly symmetric.

So let us suppose that Alpha wishes to translate Beta's arithmetical utterances and that they do so using a translation function τ_α. A first question is whether Alpha is under any obligation to translate Beta in a fine- rather than a coarse-grained way: should τ_α apply to the subsentential components of Beta's vocabulary, so that e.g. $\tau_\alpha(N_\beta)$ is a *predicate* of Alpha's language and $\tau_\alpha(0_\beta)$ is a *term* of Alpha's language? While this seems to have been assumed in the literature, it isn't obvious to us that it is something that a pluralist should accept without pause.[13]

Consider an analogy arising from the dispute between mereological universalists (who hold that a composite object exists for every plurality of objects) and nihilists (who hold that no composite objects exist). Many philosophers are attracted to the view – known as "quantifier variance" – that each side in this debate is (or could be) speaking truly in their own language – in effect, a

[12] The theorem is internal in the sense that it takes place in the object-theory, as opposed to some (perhaps second-order or set-theoretic) meta-theory. We similarly emphasize that f is an *internal* isomorphism, not an isomorphism in the set theorist's sense of a structure-preserving map between *models*.

[13] Field (2001, p. 358), for example, says this: "The question at issue is whether X must translate Y's number-theoretic vocabulary homophonically; or rather, whether X must suppose that if he introduces special terms as translations of Y's vocabulary (say 'number*', 'successor*', and so forth), he has an argument for an equivalence between his own vocabulary and the translation of Y's" – but he apparently does not note that it is a substantive assumption that the translation is fine-grained in the first place.

kind of pluralism about objecthood.[14] Nihilists can translate the utterances of universalists; for instance, they can translate universalist truths such as "there are tables" as "there are atoms arranged tablewise". But if they were forced to provide a fine-grained translation, e.g. to have a predicate P in their language for "is a table" and translate "there are tables" as "there are Ps" instead, they would be coerced in effect to import the ontology of the universalist into their own language, thereby giving up their nihilist position.

A less controversial analogy arises from the debate between classical and intuitionistic logicians. While it is natural to regard them both as speaking coherently in their own language, and translations are available on each side, coercing the intuitionist to expand their language with an operator that obeys the same principles as classical negation would immediately force their own negation to behave classically too, as Harris (1982) has shown.

At any rate, let us waive this objection and suppose that Alpha's translation is fine-grained. We can suppose further that (i) Alpha interprets Beta charitably, so that Alpha's translation of Beta's language satisfies the axioms of Beta's theory T_β, and (ii) Alpha accommodates the open-endedness of Beta's induction schema by being prepared to accept (the translation of) all instances of induction for any predicate that Beta can introduce into their language.

Even with all these assumptions the argument faces problems.[15] Alpha can introduce a function $f : N_\alpha \to \tau(N_\beta)$, defined as before. The proof that f is one-one goes through, since it appeals to an instance of T'_αs induction schema and elementary arithmetical reasoning in (Alpha's translation of) Beta's language, which is unproblematic given charitable interpretation. But the argument that f is *onto* is more problematic. As noted above, in the original, intra-language case, this used an application of T'_βs induction schema to a formula containing both f and N_α. But, in the present case, Alpha has access to T'_βs induction schema *only via translation*. So what this amounts to is supposing that *Beta* can introduce a term into their language that *Alpha* is constrained to translate as N_α. And this is not at all clear.

It might be thought that there's a natural way for Beta to introduce such a term after all. Beta is free to translate Alpha's vocabulary (let's suppose, via τ_β), and everything we've said about about charity and so on should also go for Beta's translation. So the natural candidate is $\tau_\beta(N_\alpha)$. And it is true that, if we had a good argument for $\tau_\alpha(\tau_\beta(N_\alpha)) = N_\alpha$, we would be done (actually, the weaker claim that $\tau_\alpha(\tau_\beta(N_\alpha))$ and N_α are internally isomorphic would suffice); Alpha could accept (the translation of) the relevant instance of Beta's induction schema and use it to prove that f is onto. But the problem is that if Alpha takes seriously the epistemic possibility that Beta's 'numbers' are not isomorphic to theirs, then so too should Alpha take seriously the possibility that $\tau_\alpha(\tau_\beta(N_\alpha))$ and N_α are not isomorphic. This is worth unpacking a little.

Alpha has shown that f is a structure-preserving injection from N_α to $\tau_\alpha(N_\beta)$. So there are two remaining epistemic possibilities:

(i) N_α is internally isomorphic to $\tau_\alpha(N_\beta)$;
(ii) N_α is internally isomorphic to *a proper initial segment* of $\tau_\alpha(N_\beta)$.

The possibility of scenario (ii) is the problem: put informally, that Beta's numbers are 'non-standard' or 'wider' than Alpha's; and this is exactly what would be ruled out if f could be shown to be onto.

[14] See for instance Hirsch and Warren (2019) and the references therein.

[15] The points in the next few paragraphs are directly inspired by Field (2001, Postscript to Ch 12).

Now, let's reason from Alpha's perspective. We know that Beta can run through an argument, precisely parallel to the one that we ourselves used to establish the injectivity of f, to conclude "my numbers embed into Alpha's". So we should accept the translation of this: that Beta's numbers embed into *Beta's conception of our numbers*. But in scenario (ii), where Beta's numbers are 'wider' than our numbers, this means that Beta's conception of our numbers is *also* 'wider' than our numbers. So if we cannot rule out (ii) in advance, we cannot legitimately suppose that Beta's conception of our numbers is the same as ours, since this simply fails to hold if (ii) is the case.[16]

In short: the assumption that Beta can introduce a term into their language that is internally isomorphic to N_α in this context simply amounts to the idea that f is surjective; we cannot appeal to that assumption in an attempt to establish this fact. So the attempt to deploy the categoricity argument in an inter-language context falters.

There is one more part of Parsons' discussion worth mentioning. After considering difficulties along these lines, he suggests that:

> [...] speakers take each other's words at face value and respond to them without theorizing about what they mean unless difficulties arise that such reflection could help to resolve. This would be in line with the view, which I have defended elsewhere, that language as used is prior to semantic reflection on it [...] [Alpha] will take his own number predicate as a well-defined predicate according to [Beta], and so he will allow himself to use it in induction on [Beta's] numbers. (Parsons, 2008, pp. 284-6)

The idea, we take it, is that there is a kind of default metasemantic presumption, based perhaps on the speakers' common humanity or membership of a common linguistic community, in favour of taking $\tau_\alpha(\tau_\beta(N_\alpha))$ simply to be N_α. Does this rescue the argument?

Two points are worth making about this manoeuver. First, there is a risk of undermining the reasons why a categoricity argument was needed in the first place. If we can appeal to reassuring metasemantic presumptions to get us out of trouble, why couldn't an analogously justified presumption simply rule out the possibility of non-isomorphic conceptions of the natural numbers more directly? Whether this is a real concern will depend on how exactly the presumption is defended, but without a detailed argument for it, one can be forgiven for worrying that it makes the whole discussion of categoricity an unnecessary detour.

The second, more decisive, response is that this argument simply cannot work against the kind of pluralism we are investigating. We are considering speakers who accept *apparently incompatible* arithmetical theories in their own languages, of the form $\mathsf{PA} + \varphi$ and $\mathsf{PA} + \neg\varphi$. Even if the metasemantic presumption is in good standing, it is nevertheless only a presumption, and one that can be defeated. Parsons himself is explicit that it holds "at least so long as [Alpha and Beta] do not come to disagree about the principles of arithmetic." But any instance of pluralism will involve precisely such disagreement.

We conclude that extant categoricity arguments have no purchase against arithmetical pluralism. We next consider a different kind of argument, one that appeals instead to the arithmetization of syntax.

[16] We've put this all slightly informally to aid understanding, but more formally, the situation is this – writing $\preceq$ ($\prec$) for internal isomorphism to a (proper) initial segment. Alpha has proved $N_\alpha \preceq \tau_\alpha(N_\beta)$ by the one-one step. Beta can reason analogously that $N_\beta \preceq \tau_\beta(N_\alpha)$. Alpha accepts the translation of this, i.e. $\tau_\alpha(N_\beta) \preceq \tau_\alpha(\tau_\beta(N_\alpha))$ and so Alpha has $N_\alpha \preceq \tau_\alpha(N_\beta) \preceq \tau_\alpha(\tau_\beta(N_\alpha))$. But if $N_\alpha \prec \tau_\alpha(N_\beta)$ – as possibility (ii) has it (note the proper inclusion) – then $N_\alpha \prec \tau_\alpha(\tau_\beta(N_\alpha))$ too.

3 | THE PUTNAM-KOELLNER ARGUMENT

Gödel's technique of the arithmetization of syntax allows claims about syntax – broadly understood to include notions like provability and consistency – to be expressed within arithmetic itself. This is done by setting up a coding – a systematic correspondence between strings of symbols (and sequences thereof) and numbers. For instance, given a coding, there is a sentence Con(PA) which intuitively says that there's no derivation of an inconsistency from the axioms of PA. Gödel's second incompleteness theorem tells us that if PA is consistent, it cannot prove Con(PA); a result usually understood as the inability of PA to prove its own consistency. It follows that, if PA is consistent, both PA + Con(PA) and PA + ¬Con(PA) are consistent too, so pluralists typically view both theories as true, each in their own language.

Before we consider Putnam's and Koellner's argument, let us introduce some terminology. Say that a domain is *objective* if pluralism about it fails, and that a claim or question is objective if any true theories of the relevant domain must agree on it. In this intended sense, geometric pluralists deny that there is an objective fact of the matter about whether parallel lines intersect; and arithmetical pluralists of the kind we've been considering agree that the Peano axioms are objectively true but deny the objectivity of undecidable sentences such as Con(PA). We caution that this is by no means the only possible meaning that one might attach to the (notoriously fraught and overburdened) notion of objectivity. However, it's a natural way of speaking in the context of pluralism, and one that is already widespread in the literature.[17]

In these terms, the Putnam-Koellner objection goes, syntax is a subject that deals in objective matters of fact. There is an objective answer to the question whether a certain sentence follows from certain axioms or whether a given theory is consistent. These are not matters of choice or convention. If two theories of syntax deliver different answers about whether a theory is consistent, one of them must be simply wrong.

But if there is an objective fact of the matter about whether PA is consistent, and the sentence Con(PA) expresses the claim that it is, then there must be an objective fact of the matter about Con(PA) after all: it is either true, in which case PA + ¬Con(PA) is an unsound theory, or false, in which case PA + Con(PA) is an unsound theory (indeed, an inconsistent one). Either way, one of these apparently incompatible arithmetical theories is false, and so pluralism fails.

Here is Putnam's version of the argument, directed towards the pluralist account of mathematics endorsed by the logical positivists:

> [O]nce we grant that there is at least *one* mathematical fact which is not simply our stipulation – that, for example, the *consistency* of our stipulations/practices is not itself just another stipulation or practice – then logical positivist/Wittgensteinian accounts of logical and mathematical truth are seen to be bankrupt. (Putnam, 1994, p. 501)

And here is Koellner raising a similar point against Reichenbach's conventionalist pluralism:

> This radical form of pluralism came to be challenged by Gödel's discovery of the incompleteness theorems. To begin with, through the arithmetization of syntax, the metamathematical notions that Reichenbach takes to fall within the provenance of "critical investigation" [e.g. provability and consistency] were themselves seen to be part of arithmetic. Thus, one cannot, on pain of inconsistency, say that there is a

[17] See for instance Field (2001, Ch. 11), Warren (2015), Clarke-Doane (2020a, §1.6, §6.2), Clarke-Doane (2022a, §3.5).

> question of truth and falsehood with regard to the former but not the latter. (Koellner, 2009, p. 84)

Putnam and Koellner use slightly different terminology in putting the point. Where we have spoken of the objectivity of syntax, Putnam talks of syntax not being a matter of stipulation, and Koellner talks of it being a domain about which questions of truth and falsehood arise. But we take the central shared idea to be clear: there is no space for pluralism about syntax. Faced with two theories of syntax which make apparently incompatible claims involving notions like consistency, at least one must simply be false.

An immediately salient option is for the pluralist to deny that syntax is an objective domain and to go pluralist about syntax too. We will return to this response in Section 5. But until then, we will consider approaches that try to reconcile pluralism about arithmetic with the objectivity of syntax.

One way of pursuing this response, discussed sympathetically by Clarke-Doane (2020b), is to retreat to a weaker position according to which consistency is not enough: arithmetical theories must be Σ_1-*sound*.[18] Imposing Σ_1-soundness rules out theories that get Π_1-sentences wrong.[19] It follows that theories that get consistency statements wrong are automatically excluded. For, given any (recursively axiomatizable) theory T, $\mathsf{Con}(T)$ is a Π_1-expression of the form $\forall x \neg \mathsf{Proof}_T(x, \ulcorner \bot \urcorner)$, where $\mathsf{Proof}_T(x, \ulcorner \bot \urcorner)$ is a quantifier-free formula expressing that x is a proof of some refutable sentence $\bot$ in T (given a fixed coding).

However, this proposal for rescuing pluralism is unstable. It is not only Π_1-statements that express syntactic claims. For instance, ω-consistency is a relatively natural syntactic property: it says that a theory does not prove $\exists x \neg\varphi(x)$ while proving $\varphi(\bar{n})$ for each natural number n. The simplest arithmetical sentences that express this property are Σ_2. In fact, sentences of arbitrarily high arithmetical complexity express syntactic claims – although, admittedly, some will be more natural than others. So pluralists who wish to uphold the objectivity of syntax should require not mere Σ_1-soundness but arithmetical soundness *simpliciter*, which is precisely the considered view of Clarke-Doane (2020a).[20] However, this rules out all but the one true arithmetical theory (whatever exactly it may be); and so this approach abandons *arithmetical* pluralism altogether.[21]

[18] Arithmetical sentences are classified in the arithmetical hierarchy in terms of their quantificational complexity. At the base, Π_0- and Σ_0-formulae – also known as Δ_0 – are (logically equivalent to) quantifier-free formulae. Π_{n+1}-formulae are equivalent to strings of universal quantifiers followed by a Σ_n-expression, whereas Σ_{n+1}-formulae are equivalent to strings of existential quantifiers followed by a Π_n-expression. On the present view, pluralism holds just for theories that prove only true Σ_1-claims. (Clarke-Doane actually states it in terms of Π_1-, not Σ_1-soundness, but clearly means the latter.) For an equivalent criterion that doesn't rely on an independent grasp of arithmetical truth, Σ_1-soundness can be replaced by Σ_1-consistency. A theory is Σ_1-consistent just in case $\exists v\varphi$ isn't derivable if, for every natural number n, $\neg\varphi(\bar{n})$ is provable in the theory, where φ is a quantifier-free expression. Σ_1-soundness and Σ_1-consistency are equivalent in this context, as PA decides every quantifier-free sentence (cf. Smith (2007)).

[19] Let T be an arithmetical theory and φ a Π_1-sentence. There are two ways in which T could get φ wrong: either φ is true but T proves $\neg\varphi$, or φ is false and T proves it. Recall that the negation of a Π_1-sentence is Σ_1 (cf. fn 18). So, if φ is true but T proves $\neg\varphi$, T entails a false Σ_1-statement, i.e. T isn't Σ_1-sound. On the other hand, if φ is false, $\neg\varphi$ is a true Σ_1-sentence. Since PA is Σ_1-complete – it entails all Σ_1-truths – and T extends PA, T must entail $\neg\varphi$. So, if T proves φ, it is inconsistent and, therefore, Σ_1-unsound again.

[20] Clarke-Doane's reasons for requiring arithmetical soundness are subtly different from Putnam and Koellner's. His motivation is that denying the objectivity of consistency would threaten the objectivity of certain explanations that pluralists have sometimes been inclined to offer – e.g. explanations, like that offered by Balaguer (1998, Ch. 3), of the reliability of our mathematical beliefs via the pluralist thesis that any consistent theory is true (cf. Clarke-Doane (2022a, §3.5)). The bearing of pluralism on these putative explanations is interesting but would take us too far afield to discuss here.

[21] Replacing Σ_1-soundness with Σ_1-consistency results in a more nuanced position. The argument offered in this paragraph suggests one should require not only Σ_1-consistency but Σ_n-consistency, for any natural number n. Since, for $n > 1$,

A similar response is offered by Field (2022) on slightly different grounds. Field argues that any arithmetical objectivity induced via the connection with syntax will be limited to Σ_1- and Π_1-sentences. His idea is that syntax (what he calls "protosyntax") should be formulated modally, using an operator $\Diamond B$, asserting "the possibility of whatever finite strings are needed to satisfy the protosyntactic formula B, compatibly with the laws of finite protosyntax." (Field, 2022, p. 15). But, Field argues, on this approach, it is difficult to make sense of iterated modalities, and so we end up with a theory that only interprets the restricted fragment of arithmetic just mentioned. However, Field's modal protosyntax is too austere. For one thing, it is hard to see how to formalize perfectly sensible syntactic properties like ω-consistency within it. And, more fundamentally, it seems both coherent and desirable to be able to express iterated modal claims like "it is always possible to extend a string by adjoining a primitive symbol". This suggests that a more natural modal theory of syntax will involve 'constructivist' modal operators along the lines of "no matter which strings are constructed,…" and "it is possible to construct strings such that…", and allow iterated applications of these operators.[22] But natural formulations of modal protosyntax written in these terms will interpret the whole of arithmetic, not only Field's restricted fragment.[23] So, again, if syntax is taken to be objective, there seems to be little room for arithmetical pluralism.

Fortunately for the pluralist, there is a better response, defended by Warren (2015).[24] Pluralists say that both $\mathsf{PA} + \mathsf{Con}(\mathsf{PA})$ and $\mathsf{PA} + \neg\mathsf{Con}(\mathsf{PA})$ are true arithmetical theories. Putnam and Koellner respond that, since $\mathsf{Con}(\mathsf{PA})$ expresses the consistency of arithmetic, and this is an objective matter, then at least one of these theories is objectively unsound. But, strictly speaking, the pluralist claim is that $\mathsf{PA} + \mathsf{Con}(\mathsf{PA})$ and $\mathsf{PA} + \neg\mathsf{Con}(\mathsf{PA})$ are each true *in their own language*. Making the issue of language-relativity explicit shows that, actually, the claim that $\mathsf{Con}(\mathsf{PA})$ expresses the consistency of arithmetic is not something that can be assumed to hold true across different languages whose arithmetical vocabulary satisfies different background arithmetical theories, as we'll now explain.

Putnam and Koellner's argument can be reconstructed as follows:

(1) There is an objective fact of the matter about whether PA is consistent (objectivity of syntax).
(2) So: either $\mathsf{Con}(\mathsf{PA})$ is true in every arithmetical theory or $\neg\mathsf{Con}(\mathsf{PA})$ is true in every arithmetical theory (from (1), by arithmetization).
(3) So: either $\mathsf{PA} + \mathsf{Con}(\mathsf{PA})$ or $\mathsf{PA} + \neg\mathsf{Con}(\mathsf{PA})$ is an objectively unsound arithmetical theory.

The problem with the argument, as Warren points out, is that the move from (1) to (2) is unwarranted. Following his presentation, let's dramatize the situation by supposing that we, speaking English, adopt PA, and imagining a character – the Martian – who speaks an alternative arithmetical language in which $\mathsf{PA} + \neg\mathsf{Con}(\mathsf{PA})$ is true. The reason why the step fails is that there is no guarantee that the *Martian* sentence $\neg\mathsf{Con}(\mathsf{PA})$ expresses the same syntactic content as the *English* sentence $\neg\mathsf{Con}(\mathsf{PA})$. After all, $\neg\mathsf{Con}(\mathsf{PA})$ is true in Martian but plausibly not in English,

Σ_n-consistency is weaker than Σ_n-soundness, this strategy leaves the arithmetical pluralist some room to manoeuver. Arithmetical pluralism might thus be salvaged, but it is substantially weaker and less principled. See also Berry (2023).

[22] The modality here might be taken to be metaphysical, or perhaps some kind of physical constructability (maybe abstracting from the possible finiteness of the universe).

[23] Standard formulations of non-modal syntax theory are synonymous with Peano arithmetic (cf. Corcoran et al. (1974)); and, assuming that the relevant modality satisfies S4.2, it follows by a theorem of Linnebo (2013) (Theorem 5.4) that natural protosyntax theories are also easily intertranslatable with the whole of Peano arithmetic.

[24] See also Azzouni (2023).

since Martian arithmetic contains $\neg\mathsf{Con}(\mathsf{PA})$ as an axiom while English does not (and, assuming PA is consistent, nor as a theorem). The inference from (1) to (2) requires that $\mathsf{Con}(\mathsf{PA})$ expresses the consistency of PA in every arithmetical language, but that is something we cannot assume.

Differently put, one cannot assume that the Martian sentence $\neg\mathsf{Con}(\mathsf{PA})$ should be translated homophonically into English. If it were, then the Putnam-Koellner argument would go through. But pluralists have independent reasons to think that it should not. One particularly simple way to see this is to consider the geometric analogue. If geometric pluralism is right, then Euclideans can and should view non-Euclideans as speaking truly *in their own non-Euclidean language*. But it would be a catastrophe to do this *at the same time as insisting upon a homophonic translation*: doing so would land the Euclideans with the inconsistent claim that parallel lines both intersect and do not.

In fact, from a non-pluralist, external set-theoretic perspective there are positive reasons to resist homophonic translation. Even if this external perspective ultimately cannot be accepted by pluralists, it can help us better understand why the move to (2) isn't justified. The standard non-pluralist position is that our arithmetical language has a unique intended interpretation, $\mathcal{N}$, that consists of a domain $\mathbb{N}$ (the set of natural numbers) and assigns each primitive arithmetical expression their intended meaning ("0" denotes 0, "s" picks out the successor function, etc).

PA – the arithmetic we accept – is true in $\mathcal{N}$. Martian arithmetic, by contrast, isn't. This is because PA can be shown to entail $\neg\mathsf{Proof}_{\mathsf{PA}}(\bar{n}, \ulcorner\bot\urcorner)$ for every natural number $n \in \mathbb{N}$, so every one of these sentences must be true in $\mathcal{N}$. It follows that $\neg\exists x\mathsf{Proof}_{\mathsf{PA}}(x, \ulcorner\bot\urcorner)$ – i.e. $\mathsf{Con}(\mathsf{PA})$ – must be true in $\mathcal{N}$ too, so $\neg\mathsf{Con}(\mathsf{PA})$, an axiom of Martian arithmetic, must be false. In fact, whatever structure $\mathcal{M}$ Martian arithmetic may be about, it will not be isomorphic to $\mathcal{N}$. For every instance $\neg\mathsf{Proof}_{\mathsf{PA}}(\bar{n}, \ulcorner\bot\urcorner)$ with $n \in \mathbb{N}$ will also be true in $\mathcal{M}$ but, at the same time, so will $\exists x\mathsf{Proof}_{\mathsf{PA}}(x, \ulcorner\bot\urcorner)$. In other words, Martian arithmetic is ω-inconsistent: $\mathcal{M}'$s domain is bound to have non-standard 'numbers' besides the standard ones to serve as witnesses of this existential claim.

Now the coding that allows arithmetical sentences of English to express syntactic claims maps each linguistic expression to a natural number in $\mathbb{N}$ – its code. Thus we may interpret arithmetical claims of our language, which talk about numbers in $\mathbb{N}$, as talking indirectly about the linguistic expressions (or sequences thereof) that these numbers code. For instance, if the arithmetical formula $\mathsf{Proof}_{\mathsf{PA}}(x, y)$ is true in $\mathcal{N}$ of a pair of numbers x and y just in case x is the code of a proof – a certain kind of sequence of formulae – of the sentence coded by y from the axioms of PA, we may interpret the sentence $\exists x\mathsf{Proof}_{\mathsf{PA}}(x, \ulcorner\bot\urcorner)$ – $\neg\mathsf{Con}(\mathsf{PA})$ – as saying that there is a proof of an inconsistency in PA.

In Martian, however, the coding mechanism breaks down. Arithmetical sentences in Martian don't talk exclusively about numbers in $\mathbb{N}$. As we said, in every model of Martian arithmetic $\mathsf{Proof}_{\mathsf{PA}}(x, \ulcorner\bot\urcorner)$ will be true only of non-standard numbers, that is, other than codes of proofs. Thus, $\exists x\mathsf{Proof}_{\mathsf{PA}}(x, \ulcorner\bot\urcorner)$ cannot correctly be interpreted as expressing the inconsistency of PA, as the non-standard elements that witness the existential claim cannot be regarded as coding up genuine proofs.

The last few paragraphs assumed a non-pluralist, external perspective to argue that the Martian sentence $\neg\mathsf{Con}(\mathsf{PA})$ does not express the inconsistency of PA. As we argued above, the pluralist already has reasons for scepticism about this move. To what extent can this case be supplemented by the just-mentioned considerations? In fact, the considerations about 'standard' and 'non-standard' models transpose, in a way. While pluralists cannot appeal to a transcendental notion of *the standard model* of arithmetic, they can appeal to a notion of *number-system isomorphic to mine* that plays much the same role. When confronted with a Martian who accepts an axiom like ¬Con(PA) that we do not, there is *at least* an epistemic possibility (for much the same reasons as those discussed in Section 2) that the 'numbers' of this deviant theory (i.e. our translation of the Martian's number predicate) are not isomorphic to ours; in which case, a homophonic translation is blocked.

Summing up: there are strong reasons for pluralists and non-pluralists alike to deny that the Martian sentence ¬Con(PA) expresses the inconsistency of PA, and so no clear argument against pluralism to the effect that it gets consistency facts wrong. As Warren puts it: "it isn't that the [Martian] theory proves claims that are false in the language of the theory, rather, it is that the theory proves true claims that don't express or mean what the relevant syntactic facts mean." He goes on to conclude that "it is coherent to suppose that consistency is a matter of fact, while arithmetical truth, even for consistency sentences, is a matter of convention" (Warren, 2015, pp. 1364-5).

But, we believe, his conclusion is premature. There is a better argument against arithmetical pluralism – one that really does refute the combination of arithmetical pluralism and objectivism about syntax. Our argument, to which we turn, appeals to a Parsons-style categoricity result. But unlike Parsons' argument, ours doesn't purport to show that any two arithmetics must be isomorphic. It shows instead that there's an (internal) isomorphism between arithmetic and syntax. So, if arithmetical pluralism is correct, the objectivity of syntax must be given up.

4 | A NEW ARGUMENT FROM CATEGORICITY

The flaw in the Putnam-Koellner argument is that, while our coding allows sentences in the arithmetical fragment of *English* to express syntactic claims, there is no guarantee that any coding will continue to function as expected in 'alien' arithmetical languages. In other words, there is a gap between arithmetic and syntax in alternative arithmetical theories. However, as we will now argue, an internal categoricity result can bridge this gap.

When we are being careful about the interaction between syntax and arithmetic, we should recognize the disentanglement of the two: that is, our theory of syntax should be formulated in a distinct language which talks directly about genuinely syntactic objects, not numbers.[25] So we need two things: a language of syntax – in which claims like "such-and-such a string is a well-formed formula of the language of arithmetic" and "PA is consistent" can be formalized – and axioms stated in that language, expressing basic truths about its notions. There are a few choice-points here.

A first question is how to think about the ontology of syntax: this might be either possible concrete string-inscriptions or abstract string-types; we will assume the latter, for convenience, though everything we say carries over to the former (at the cost of some unnecessary complexity).

Then there is the question of which syntactic notions to take as primitive. One approach starts with the elementary symbols of the object language and a primitive operation of concatenation of strings; another starts with the empty string and, for each elementary symbol, an operation of adjoining that symbol to a string. But the choice doesn't matter, as it turns out that the natural ways to axiomatize these two approaches are equivalent, and both are indeed equivalent to Peano arithmetic in the precise sense of being synonymous or mere notational variants.[26] So there is no

[25] For more on disentanglement, see Leigh and Nicolai (2013), Heck (2015), Mount and Waxman (2021).

[26] See Corcoran et al. (1974) for the details of the equivalence for syntax theories with more than one symbol.

harm or generality lost (and some convenience gained) in thinking of the language of the syntax theory as a disjoint copy of the language of arithmetic – as Heck (2015, p. 451) puts it, as "the language of arithmetic written in boldface".

For every purely arithmetical formula φ, we can define φ^{S} as the result of replacing in φ each primitive arithmetical symbol with its syntactic counterpart. So we assume that the syntax theory extends PA^{S}, the theory whose axioms are the syntactic counterparts of each of the PA axioms. So interpreted, the axioms express syntactic truths like "for every string *s*, the result of appending a single symbol to *s* is a string", "the empty string is not the result of appending a symbol to any string", and so on.[27] All of this, we emphasize, is unmediated by any coding: the *intended* interpretation of the syntax language is syntactic, not numerical.

So let T be a (purely) arithmetical theory (i.e. an extension of PA), ST a syntax theory that extends PA^{S}, and $\mathsf{NS}(T)$ the combined theory where both arithmetical and syntactic induction schemata are extended to include vocabulary of the combined language. We can then appeal to an intra-linguistic application of Parsons' internal categoricity result discussed in Section 2 to conclude that there is an internal isomorphism $f : N \to N^{\mathsf{S}}$ between T and ST, from which it follows that:[28]

$$\mathsf{NS}(T) \vdash \varphi \leftrightarrow \varphi^{\mathsf{S}}$$

In other words, if we combine an arithmetical theory with our theory of syntax, and the induction schema of each is understood open-endedly, then arithmetical and syntactic claims must co-vary in truth-value.

Let f serve as our coding, and Con(PA) – formulated in the language of T – be the purely arithmetical sentence expressing the consistency of PA relative to f. It follows that the purely syntactic statement $\mathsf{Con}(\mathsf{PA})^{\mathsf{S}}$ – formulated in the language of ST – *directly* expresses the claim that PA is consistent, unmediated by any coding. The result above shows that it is a theorem of the combined theory of arithmetic and syntax that

$$\mathsf{Con}(\mathsf{PA}) \leftrightarrow \mathsf{Con}(\mathsf{PA})^{\mathsf{S}}$$

Put somewhat more evocatively: mild assumptions about arithmetic and syntax force an equivalence between the *syntactic* consistency sentence and the *arithmetical* consistency sentence. This is the central fact that can be used to plug the gap in the Putnam-Koellner argument. Here is how it goes.

Return to the two arithmetical theories $\mathsf{PA} + \mathsf{Con}(\mathsf{PA})$ and $\mathsf{PA} + \neg\mathsf{Con}(\mathsf{PA})$. If arithmetical pluralism is right, each of these theories is true in some language. Since we are being careful

[27] In assuming that any acceptable syntax theory must extend (a notational variant of) Peano arithmetic, we are ruling out the analogue of strict finitism about syntax (though we don't know of any published proposal to this effect). Our response to a challenge here would be similar to fn 7. Even if strict finitism about syntax is coherent, there is interest in exploring the consequences of infinitary syntax theories. And there are also reasons, perhaps even stronger than in the arithmetical context, for thinking that strict finitist theories cannot play the relevant conceptual role (for instance: it is difficult to see how to theorize about the syntax of Peano arithmetic – a non-finitely axiomatizable theory – or to investigate questions of its consistency – questions concerning the infinitely many possible proofs – in the setting of a strict finitist syntax theory). Thanks to a referee for discussion.

[28] The proof is entirely parallel to Parsons', where PA and ST play the role of T_α and T_β. In our setting, one of the 'copies' of arithmetic is really a theory of syntax; but of course all that's formally necessary is that the theory satisfies the axioms – its interpretation plays no role in the argument.

about distinguishing syntactic from arithmetical matters, assume that these languages each have a disentangled syntax theory as above. The categoricity result shows that each of the combined theories proves the biconditional $\mathsf{Con(PA)} \leftrightarrow \mathsf{Con(PA)}^{\mathsf{S}}$. In one language the *arithmetical* sentence $\mathsf{Con(PA)}$ is true (and provable), and so, via categoricity, so too is the *syntactic* sentence $\mathsf{Con(PA)}^{\mathsf{S}}$; similarly $\neg\mathsf{Con(PA)}^{\mathsf{S}}$ is true in the other language. But then there are at least two apparently incompatible theories of *syntax* that are each true in their own language. In other words, the pluralist about arithmetic who wishes to maintain the objectivity of syntax is in trouble: pluralism about arithmetic leads to pluralism about syntax. Here is the argument more explicitly:

(1) Each of $\mathsf{PA} + \mathsf{Con(PA)}$ and $\mathsf{PA} + \neg\mathsf{Con(PA)}$ is true in some arithmetical language (pluralism about arithmetic).
(2) So: each of $\mathsf{Con(PA)}^{\mathsf{S}}$ and $\neg\mathsf{Con(PA)}^{\mathsf{S}}$ is true in some *syntactic* language (from (1), by categoricity).
(3) So: there are two true but apparently incompatible theories of syntax (from (2)).

The problem with the Putnam-Koellner argument, as we saw, is that the arithmetization of syntax does not allow us to understand $\neg\mathsf{Con(PA)}$ as expressing the inconsistency of PA *in Martian*. By contrast, our argument goes via sentences of the pure syntax language (which by construction say directly that PA is inconsistent, unmediated by any coding) together with an application of the categoricity result discussed above. Even though the Martian arithmetical sentence $\neg\mathsf{Con(PA)}$ doesn't directly express the inconsistency of PA, categoricity entails that $\neg\mathsf{Con(PA)}$ is *provably equivalent* to a sentence in the Martian pure syntax language that does.

Let's take a closer look at some issues that arise for the inference from (1) to (2), the crucial move in the argument. The step is justified in large part by categoricity. But notice that it subtly requires that the categoricity result holds internally in all of the relevant theories we are considering, including those with a 'deviant' arithmetical component. So it is worth verifying that speakers of alternative arithmetical languages, such as our Martians, satisfy its conditions.

We've already argued that it's appropriate to assume disentangled theories of arithmetic and syntax (and, surely, any pluralist who seeks to disarm the original Putnam-Koellner argument by appealing to a sharp distinction between syntax and arithmetic is in no position to disagree). And we've also argued that arithmetic and syntax theories can be assumed to each extend PA (in their respective vocabularies). There are two slightly more substantive philosophical points to defend before the categoricity theorem can be invoked: first, our attribution to the Martians of a *combined* theory, incorporating both syntactic and arithmetical components, and second, the assumption that both induction schemata must be understood open-endedly (so that syntactic vocabulary can feature in instances of arithmetical induction and vice versa). Let us take each in turn.

Could it be maintained that although Martians accept both an arithmetical theory and a syntax theory, there's some principled reason not to attribute them the combination of the two? This is not an appealing avenue of response. For it to be plausible, presumably, we'd have to think of the Martians as maintaining a kind of strict compartmentalization between their syntactic theorizing and their mathematical theorizing. Now, of course, Martians are fictional creatures, brought in to make vivid the idea of an alternative linguistic practice just like ours except for slight differences concerning purely arithmetical axioms. But the more we flesh out the story to include compartmentalization of arithmetical and syntactic vocabulary, the further such a practice can be seen to depart from our own. We maintain no such separation: we happily make sense of, e.g. mixed conjunctions or biconditionals between syntactic and arithmetical claims. Pluralism is stated in terms of theories whose conceptual role is sufficiently similar to ours. And although this is somewhat vague and context-sensitive, it is clear enough that it rules out theorizing about syntax and arithmetic in such strict isolation from one another. As we emphasized earlier, a core part of the conceptual role of arithmetic involves non-arithmetical applications, and mixed theorizing is essential to these. For instance, speaking of even such a basic notion as the *length* of a string unavoidably involves the entanglement of arithmetical and syntactic vocabulary.

Extended induction is, similarly, a crucial tool for applying arithmetic to other domains. For example, consider how we prove the deduction theorem for, e.g. first-order logic. We show the result holds for one-line proofs; we show that the result holds for proofs of a certain length provided it holds for proofs with fewer steps; and we conclude that the result holds for proofs of every length *by induction on the length of proof*. But notice that this use of induction essentially involves an instance in the extended language, containing not just arithmetical but also syntactic vocabulary. And there is nothing special about syntax here; this kind of inference is deeply embedded in our applied arithmetical reasoning. As McGee notes: "We don't have to reassess the validity of mathematical induction when we expand our inventory of theoretical concepts, for our current understanding of the natural numbers ensures that, no matter how we expand our language in the future, induction axioms formulated with the enlarged vocabulary will still be true."[29] Extending syntactic induction is just as natural. Consider again *length*. Thought of as a function from strings to natural numbers, *length* is most naturally defined by recursion (the length of the empty string is 0; if a string has length n, then adjoining an additional symbol results in a string of length $n + 1$). To prove that such a definition is legitimate, we need an instance of *syntactic* induction extended to include arithmetical vocabulary.[30]

The move from (1) to (2) also involves the assumption that, even in a language with 'deviant' arithmetical vocabulary like Martian, there can be a purely syntactic sentence $\mathsf{Con}(\mathsf{PA})^{\mathsf{S}}$ that expresses the consistency of PA. This raises a potential worry: why should there be a fragment of Martian that can express syntactic notions at all? Why should the 'syntactic fragment' of Martian be interpreted as talking about syntax in the first place?

The short answer is that if the ability of Martian speakers to adopt a syntax language is denied, this threatens to undermine the applicability of arithmetic. The concern is, in effect, that although there are true but apparently incompatible arithmetical theories, not all of these can coherently coexist with a genuine syntax theory. But if true this means that anyone who adopts a 'deviant' arithmetical practice would be simply blocked from theorizing about syntax.

The situation can be dramatized by a slight variation of the Martian thought experiment. Suppose that, at first, Earthlings and Martians constitute a single linguistic community, speaking a common language in which common syntactic – but no arithmetical – theories are formulated. Now suppose that the community decides to expand the language with an arithmetical theory in order to theorize, among other things, about the length of syntactic expressions. However, fragmentation occurs: while we Earthlings opt for PA, the Martians adopt a 'deviant' arithmetical

[29] Thinking of induction as open-ended is compatible with *some* restrictions on the range of predicates that figure in it. For instance, it's arguably well-motivated to deny instances of induction involving vague predicates, since induction is classically equivalent to the least number principle and thus entails things like "there is a least number of pennies that makes one rich". Similarly, we might refrain from extending induction to non-well-behaved fragments of our language – those which involve "inconsistent rules or other confusions" (Warren, 2017, p. 91). But it's hard to see how the cases of arithmetical and syntactic vocabulary that we're considering could be argued to possess any such defects; while of course a pluralist might try to block our argument by imposing restrictions on extended induction, this seems an *ad-hoc* and unnatural move, especially in the absence of any independent reasons to do so.

[30] In effect, syntactic induction codifies our understanding that every string can be obtained by starting with the empty string and adjoining finitely many symbols.

theory like PA + ¬Con(PA). According to the response under consideration, the Martian adoption of this *arithmetical* theory means that they are no longer capable of talking about *syntax* – their (putatively and formerly) syntactic vocabulary can no longer properly be understood as constituting a genuine syntax theory. What's more, no alternative fragment of the Martian theory could possibly play the syntax role since the categoricity result would also hold for it and so it would suffer the same fate. The Martian adoption of a deviant arithmetical theory means that they cannot, in effect, have *any syntax theory whatsoever*. But this result is an embarrassment to pluralism. Martian arithmetic is supposed to be a genuinely arithmetical theory – one that has a conceptual role similar to the role that arithmetic plays for us. And, as we've emphasized, one of the core aspects of the conceptual role of arithmetic is its capacity for (non-arithmetical) applications. And yet, according to this response, speaking a perfectly legitimate arithmetical language obliterates one's ability to express syntactic facts. It seems to us that this cannot be accepted by arithmetical pluralists: if adopting one of PA + Con(PA) or PA + ¬Con(PA) precludes theorizing about syntax, then these theories cannot both play the conceptual role of arithmetic.

So, we conclude, the new argument is sound. The bridge between arithmetic and syntax provided by categoricity means that arithmetical pluralists cannot maintain the objectivity of syntax.

5 | PLURALISM ABOUT SYNTAX?

In this final section, we briefly consider implications of our argument. The only real way out, it seems, is for pluralists about arithmetic to adopt pluralism about syntax too. So the question is whether this is an appealing or even coherent view.

As we said at the outset, pluralism about syntax seems highly counterintuitive. Here is Warren giving expression to this thought:

> Whether or not a certain sentence is derivable in a given formal system is a matter of fact. Similarly, whether or not a given sequence of expressions is a sentence (according to our grammatical conventions) is a matter of fact. In general, syntax concerns matters of fact that can't be made true or false simply by convention or stipulation. And this is true even if the rules of proof and grammar are themselves a matter of convention – cf. the rules of chess are conventional but whether or not a mate is possible with two knights and a king is a matter of fact. (Warren, 2015, p. 1357)

This certainly has some appeal. But, on reflection, we are not sure that pluralism about syntax is any *more* counterintuitive or problematic than pluralism about arithmetic was to begin with. (Of course, one might conclude: so much the worse for arithmetical pluralism).

A first point is that pluralism about syntax, just like pluralism about arithmetic, doesn't mean that anything goes. In fact, if (as we've argued) any syntax theory will contain at least a theory synonymous with PA, that is itself enough to settle many syntactic questions. Any syntactic property that can be recursively checked – including being a sentence, an axiom or proof (of a specified recursive theory) – will be decidable. Furthermore, since PA is Σ_1-sound and Σ_1-complete (cf. fn 19), its syntactic analogue will prove all and only true sentences of the form "x is provable", i.e. "there exists y such that y is a proof of x". This takes at least some of the sting from Warren's concerns: any reasonable pluralism about syntax will in fact resolutely settle questions like whether or not a given expression is a sentence or whether or not a given sequence constitutes a proof. Generally, it will settle all claims involving only bounded quantification. So, the picture must be, if there are questions over which true syntax theories 'diverge', they are inherently tied up with quantification over the infinitely many syntactic entities. This is not so different in motivation from what arithmetical pluralists are already prepared to say.

More generally, many of the motivations for pluralism about arithmetic carry over. For instance, recall the naturalistic worries that arise in explaining how the content of arithmetical language could be determinate: on their face, these arise to just the same extent with syntax. So again, the analogy seems strong.

One possible source of resistance is that, as Parsons puts it, syntax is quasi-concrete: closely tied to concrete string-inscriptions. (In spelling this out it might matter whether we think of the subject matter of syntax as being about abstract string-types or possible concrete string-inscriptions; but even on the former view, string-types are *instantiated* by concrete tokens). Perhaps this quasi-concreteness can play a metasemantic role, providing a way of 'anchoring' the content of syntactic notions that doesn't arise for arithmetic? The plausibility in this suggestion comes from the fact that interaction with concrete strings is arguably an essential part of the way we learn syntactic language and concepts. But actually, again, it is not so clear that this provides a disanalogy: after all, we start to learn arithmetical language and concepts, too, by interacting with sets or pluralities of concrete things; some (e.g. Hilbert) even identify numbers with string-types.[31] Note, moreover, that we can only encounter finitely many strings (be they string-types or concrete string-inscriptions), so whatever anchoring the quasi-concreteness of syntax may provide will plausibly be limited to the content of concepts concerning *finitely* many strings. So it is hard to see how it might help us to, for example, fix the content of the concept of *string* itself in such a way that what we may call from an external perspective "non-standard strings" are excluded, or to establish the truth of novel complex syntactic claims involving universal quantifiers (e.g. Π_1-claims), ranging over infinitely many strings. But these claims over which syntax theories might diverge (if pluralism about syntax is true) are precisely analogous to the kinds of failures of objectivity that pluralists about arithmetic are already prepared to accept.

All of this is to say that there is a stable pluralist position. As we have argued, it cannot be one that maintains that syntax constitutes an objective domain of facts. Rather, in light of categoricity, the picture has to be something like this: pluralism about arithmetic is committed to pluralism about syntax too. What's more, the two kinds of pluralism here should not be viewed in isolation: categoricity constrains one's arithmetical theory and one's syntactic theory to co-vary with one another. So, pluralism then becomes the view that there are distinct *packages* of arithmetical plus syntactic principles to choose from, all of which are true in their respective languages. Whether or not this view is ultimately attractive, and even if not, whether it is one that can be avoided, are questions for further work.[32]

[31] Part of the attraction for thinking that abstract string-types are more metasemantically tractable is arguably that they resemble the tokens that instantiate them (and, more generally, possess many properties usually distinctive of *concreta*). However, see Clarke-Doane (2022b) for arguments against this idea.

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